

---

# Spontaneous Breaking of Space-Time Symmetries

Eliezer Rabinovici

Racah Institute of Physics, The Hebrew University of Jerusalem, 91904, Israel.  
`eliezer@vms.huji.ac.il`

## Abstract <sup>1</sup>

Kinematical and dynamical mechanisms leading to the spontaneous breaking of space-time symmetries are described. The symmetries affected are space and time translations, space rotations, scale and conformal transformations. Applications are made to solidification, string theory compactifications, the analysis of stable theories with no ground states, supersymmetry breaking and the determination of the value of the vacuum energy.

---

<sup>1</sup> Based on the contribution to "String Theory and Fundamental Interactions. Celebrating Gabriele Veneziano on his 65th Birthday", edited by M. Gasperini and J. Maharana, to be published by Springer, and Lecture given at Les Houches summer school: "String Theory and The Real World From Particle Physics to Astrophysics", Session LXXXVII, 2007.

## 1 Introduction

This being a contribution to honor Gabriele Veneziano, I allow myself to open with some personal words. I have first heard Gabriele's name on the radio when the late Yuval Ne'eman described the great importance of young Gabriele's work. That was in the late sixties, several years later as a student I had the privilege to learn from a still very young Gabriele about the dual model in full detail. These were outstanding lectures. Over the years I have learned many more things from Gabriele some of it through direct collaborations, in parallel we had developed a personal friendship for which I am grateful.

It is not uncommon for young scientists to complain that their teachers didn't educate them appropriately and did not really pass them/point them to the relevant information. I may have some such complaints of my own but not to Gabriele. I would have liked for example to know earlier about the ideas of Kaluza and Klein. So in order to somewhat reduce the complaints that will be directed at me, I would like to use this opportunity to describe something that it is not taught extensively in particle physics courses, mechanisms to spontaneously break space-time symmetries. The world around us is actually neither explicitly invariant under translations, nor under rotations. It also is not explicitly invariant under scale and conformal symmetries. In this work we will review various mechanisms to break all these space-time symmetries. I think they may yet play an important role in particle physics as well. I will first describe attempts to break translational invariance kinematically by imposing specific boundary conditions. Then I will review the Landau theory of solidification and an attempt to apply it to generate a dynamical mechanism for compactifications. I will discuss both the success and challenges of that approach. Next, in the context of breaking time translational invariance, I will discuss various systems which are well defined but have no ground state. Following a review of the breaking of scale invariance and conformal invariance, I will also not miss this opportunity to describe in a Katoish manner that the vacuum energy in conformal/scale invariant theories is very constrained, and its zero value does not depend on the presence or absence of any spontaneously generated scales. This may eventually be recognized as an important ingredient in understanding and explaining the cosmological constant problem.

## 2 Spontaneous Breaking of Space Symmetries

Space symmetries include space translations and space rotations, we address the spontaneous breaking of these space symmetries. This occurs for example when a liquid solidifies and a lattice is formed. The standard manner to identify the ground state of a system is to construct what is called the effective potential. The symmetry properties of the ground state determine whether a spontaneous breaking of symmetries which are manifest in the Lagrangian occurs.

Let's review the manner in which the effective potential is constructed. One first considers all wave functionals which have the same expectation value of the field operator  $\hat{\phi}$

$$\langle \Psi(\phi) | \hat{\phi} | \Psi(\phi) \rangle = \tilde{\phi} . \quad (1)$$

Out of these subset of wave functionals, one chooses that particular wave functional which minimizes the expectation value of the Hamiltonian. One calls it  $V_{eff}(\langle \phi \rangle)$

$$V_{eff}(\tilde{\phi}) = \min_{\tilde{\phi}} \langle \Psi(\phi) | \hat{H} | \Psi(\phi) \rangle . \quad (2)$$

Eventually one draws a picture portraying  $V_{eff}$  as a function of  $\tilde{\phi}$  and one searches for its minimum. The wave functional for which this energy minimum was obtained is the wave functional of the ground state of the system. However, one usually ignores the possibility that the ground state wave function would correspond to a non-constant(in  $x$ ) expectation value  $\langle \phi(x) \rangle$ . Of course it makes much easier the drawing of pictures in books, here however we will discuss cases where  $\langle \phi(x) \rangle$  actually does depend on  $x$  when evaluated in the ground state.

Why does one usually only consider wave functionals with constant values of  $\langle \phi(x) \rangle$ ?

The reason is expediency, when one wants to pick up the ground state of the system among various candidates, one is interested only in the winner, that is the true ground state. One does not care about missing out candidate states whose energies are just above that of the ground state of the system. As generally spontaneous breakdown of space-time symmetries in the ground state is not expected, one considers it enough to search for the ground state only among those candidates for which  $\langle \phi(x) \rangle$  is constant. However that need not always be the case.

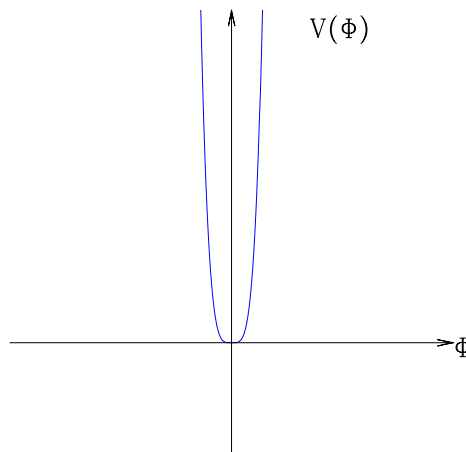
### Kinematics: Attempts to Break Spatial Translational Invariance Through Boundary Conditions

I will first describe an easy way to attempt to break space symmetries. That is to break the symmetries not by the dynamics of the system but kinematically, by imposing certain boundary conditions. This easy solution, is a mirror to what is done in String Theory in several cases, including when one is considering brane sectors. To try and break translational invariance by boundary conditions one considers for example a system which depends on a scalar field  $\phi$ . Assume the system lives in a box extending from  $-L$  to  $L$ , and impose the condition of anti-periodicity, namely

$$\phi(L) = -\phi(-L), \quad (3)$$

where  $L$  is the spacial cutoff we put on the system.

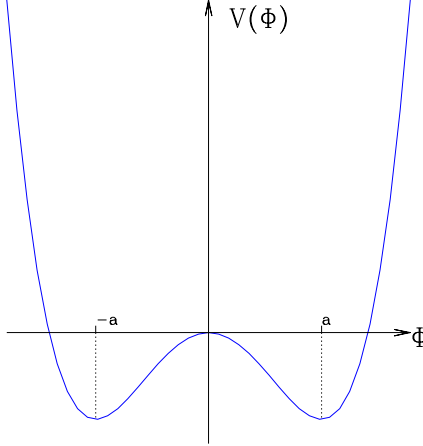
If the system at hand is described by an effective potential that has only one minimum (see fig. 1),



**Fig. 1.** Unbroken symmetry

where there the expectation value  $\langle \phi \rangle$  vanishes, then there is no effect resulting from imposing the boundary conditions. The ground state does fulfill the boundary condition, and it remains the one which does not break translational invariance. From the point of view of the wave functional, it is concentrated around  $\phi = 0$ .

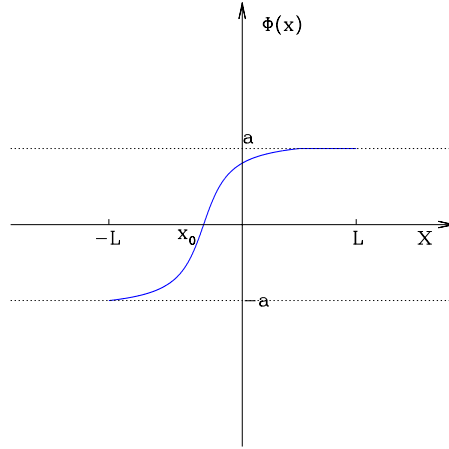
However, consider the double well potential, (fig. 2), (in circumstances where there is no tunneling).



**Fig. 2.** Broken symmetry

In this case the effective potential has two minima, one at  $\phi = a$  and the other at  $\phi = -a$ . Imposing the boundary condition removes both of the possible true vacua of the system, because neither the ground state for which  $\langle \phi \rangle = -a$ , nor the ground state for which  $\langle \phi \rangle = a$  obey the boundary condition. One is driven to look for another type of ground state. We know, for example, that in a two dimensional system composed only of scalar fields there is a finite energy solution, which is a soliton that at  $L$  has a value  $a$  and at  $-L$  has a value  $-a$ , see fig. 3. An anti-soliton will have the opposite values. This is a stable topological configuration, and one may imagine that indeed in such a system there is no translational invariance, because the ground state will have to be such that its spacial expectation value follows the values of the soliton field, and thus is not translational invariant.

It is true that by imposing the boundary conditions one has forced the system into the soliton sector, but one has to remember that this system has a zero mode. Technically, if one solves the small fluctuations of the scalar field in the presence of a background, which is a soliton, one finds that there is a zero mode. This zero mode is a reflection of the underlying translational invariance and it actually tells that one is not able to determine, by energetic considerations, where the inflection point  $x_0$ , the point from which one turns from one vacuum to the other, (see fig. 3) occurs. Actually there is a valid soliton solution for each value of  $x_0$ .



**Fig. 3.** A soliton attempts to break translational invariance.

Why is this important? At the case at hand, the zero mode is normalizable. This amounts to saying that the soliton mass is finite. In such a case, there is actually no bulk violation of translational invariance. What one needs to do is to construct an eigenstate configuration, which is an eigenstate of the linear momentum operation, a plane wave in terms of the center of mass coordinate of the soliton. The lowest energy state which corresponds to a momentum state has  $p = 0$ , one has restored in this way translational invariance. The only problem will be to fix the system very near the edges, but in the bulk, the symmetry has been restored and there is no breaking of bulk translational invariance. Could one still have a case where the boundary conditions do cause a spontaneous breaking of translational invariance?

This may occur when one drives the mass of the soliton to infinity by an appropriate choice of parameters. When the mass of the soliton is infinite, physically one cannot form a linear momentum state out of it, and technically the zero mode ceases to be normalizable. In such a case, one does indeed break translational invariance spontaneously by fixing the point where the soliton makes the transition from one vacuum to the other. This occurs for example in String Theory in a sector containing infinite branes, branes which have finite energy do not break translational invariance, and one can build out of them linear momentum states. However, branes which extend up to infinity carry infinite energy, and therefore do lead to the breakdown of translational invariance.

I will mention at this point that once upon the time, when people were considering the breakdown of extended global supersymmetries, there was a

predominant common wisdom which claimed that one cannot break down extended global supersymmetry to anything but  $\mathcal{N} = 0$ . That is either all the supersymmetries are manifest together, or they are all broken together. The argument went in the following way: one writes the formula for the Hamiltonian

$$H = \sum_{\alpha} \bar{Q}_{\alpha}^I Q_{\alpha}^I, \quad (4)$$

where  $I = 1, \dots, \mathcal{N}$  is *not* summed, it is a non-trivial constraint to get the same Hamiltonian by summing over different supersymmetry generators. When one can do that one has an extended supersymmetry. However it is clear from this that if the Hamiltonian does not vanish on the ground state, then some of the  $Q^I$  (for each  $I$  independently) do not annihilate the ground state. Therefore the supersymmetries are either all preserved or all broken.

This type of argument assumed implicitly that Poincaré invariance is present in the system. If one now considers other systems, (see for example [1], [2]), where part of the Poincaré invariance is preserved and part is broken, this exposes a loop hole in the former argument. In the absence of full translational invariance (due to the presence of infinite mass branes) one may obtain fractional BPS states, and one may break  $\mathcal{N} = 4$  down to  $\mathcal{N} = 2$ ,  $\mathcal{N} = 2$  to  $\mathcal{N} = 1$  and various other combinations.

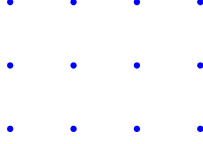
This is an example where spontaneous breaking of translation invariance occurs, it has an impact also on the partial breaking of global supersymmetry and, if one wishes, this is a way to break translation invariance by forcing the system, using boundary conditions, to a certain super-selection sector.

This is not what I mainly want to discuss here. I would like to discuss a situation where the dynamics of the system drives the spontaneous breaking of translational and rotational invariance.

## Dynamics: The Landau Theory of Liquid-Solid Phase Transitions

Let us now turn to discuss the transition between a liquid and a solid. This follows the seminal work of Landau [3]. In a monumental paper, he described spontaneous symmetry breaking of both, internal and space-time symmetries. Consider a liquid, a system whose lagrangian is either relativistic or non-relativistic, and it possesses full rotational and translational invariance. On the other hand, a solid is a system which maintains only a very small part of the translational invariance and rotational invariance, (fig. 4).

Let us simplify the study by ignoring the point structure at each lattice point which a solid may have. That is let's not consider the atomic structure



**Fig. 4.** The solid lattice breaks most of the translational and rotational invariances

at each point. One focuses first on the question of how does the simplest lattice forms.

I will describe this following Landau and then, following [4], I am going to describe applications to String Theory. Landau starts by defining the Landau order parameter to monitor the transition between a solid and a liquid. It is a scalar order parameter  $\varrho(\mathbf{x})$

$$\varrho(\mathbf{x}) = \varrho_s(\mathbf{x}) - \varrho_0, \quad (5)$$

the difference between the non-translational non-rotational invariant density of the solid  $\varrho_s(\mathbf{x})$ , and the constant density  $\varrho_0$  of the liquid. Next, consider the Fourier decomposition of  $\varrho(\mathbf{x})$

$$\varrho(\mathbf{x}) = \sum \varrho(\mathbf{q}) e^{i\mathbf{q} \cdot \mathbf{x}} + h.c. \quad (6)$$

It is useful to use as order parameters the Fourier components  $\varrho(\mathbf{q})$ .

The question is thus: Does the wave functional of the ground state have support on  $\mathbf{q} \neq \mathbf{0}$ ? If the answer is positive, then at the very least continuous translational space symmetry would be spontaneously broken. This will be determined by studying the Landau-Ginsburg effective action as expressed in terms of the order parameter  $\varrho(\mathbf{q})$ . The first relevant term of the Landau-Ginsburg action is quadratic in the order parameter and is given by:

$$\mathcal{L}_0 = \int d\mathbf{q}_1 d\mathbf{q}_2 \varrho(\mathbf{q}_1) \varrho(\mathbf{q}_2) A(|\mathbf{q}_1|^2) \delta(\mathbf{q}_1 + \mathbf{q}_2). \quad (7)$$

The delta function  $\delta(\mathbf{q}_1 + \mathbf{q}_2)$  enforces translational invariance, while rotational invariance is preserved by the dependence on  $|\mathbf{q}|^2$  of the function  $A(|\mathbf{q}|^2)$ . The function  $A(|\mathbf{q}|^2)$ , like in any Landau-Ginsburg potential, is determined by the microscopic theory. In the particular case at hand, it will depend on the hardcore potential component in the atoms involved and on

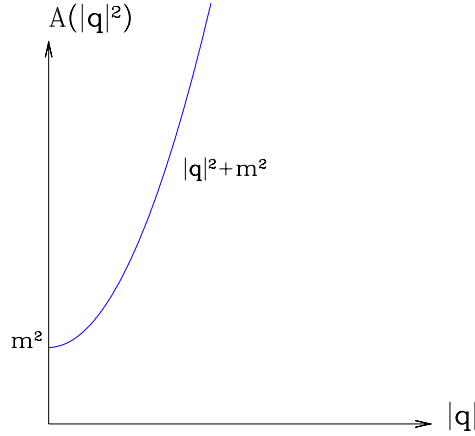


other possible potentials, as well as on the temperature of the system. In the case of neutron stars, studied in [5], the Pauli exclusion principle plays a role in determining the function  $A(|\mathbf{q}|^2)$ .

Let us treat first an example that we are familiar with, that of a free massive spin-zero particle in a relativistic field theory. In that case the function  $A(|\mathbf{q}|^2)$  is

$$A(|\mathbf{q}|^2) = |\mathbf{q}|^2 + m^2. \quad (8)$$

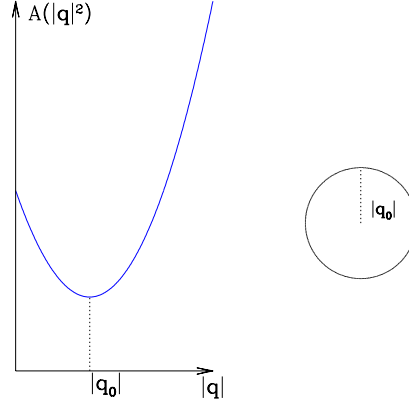
This has a minimum at  $|\mathbf{q}|^2 = 0$ , as shown in fig. 5, and thus the function  $\varrho(\mathbf{q})$  should get the support only at  $\mathbf{q} = \mathbf{0}$ , there is no spontaneous breakdown of translational invariance in this case.



**Fig. 5.** The form of  $A(|\mathbf{q}|^2)$  in a free massive relativistic field theory does not lead to spontaneous breaking of translational invariance.

In the presence of interactions things may become more complicated, for example I am not familiar even with a proof that the Standard Model ground state does not violate space-time symmetry, (though most likely it does not). In any case the microscopic theory may allow a different function for  $A(|\mathbf{q}|^2)$ . In particular, assume that the form of  $A(|\mathbf{q}|^2)$  is as given in fig. 6. In this case, the function  $A(|\mathbf{q}|^2)$  has a minimum at a value  $|\mathbf{q}_0|^2 \neq 0$ . In such a system the ground state wave functional gives rise to a density concentrated around  $|\mathbf{q}_0|^2 \neq 0$ . In particular, one would expect the support to be concentrated around a sphere in  $\mathbf{q}$ -space, whose radius is  $|\mathbf{q}_0|$ . So one is in a situation, given

$A(|\mathbf{q}|^2)$  of that form, where one has a spontaneous breaking of translational invariance, but not yet also a breaking of rotational invariance, which is what is needed to form a solid. It is good enough to break just translational invariance.



**Fig. 6.** Example of a function  $A(|\mathbf{q}|^2)$  which leads to the breaking of translational invariance. An explicit microscopical realization of a such a form appears in neutron stars [5]. The wave functional is concentrated at most on the shell of a sphere of radius  $|\mathbf{q}_0|$ .

The ground state density does depend on  $\mathbf{x}$

$$\varrho(\mathbf{x}) = \int_{S_{|\mathbf{q}_0|}} d\Omega \varrho(\mathbf{q}) e^{i\mathbf{q} \cdot \mathbf{x}} + h.c. . \quad (9)$$

In this approximation the wave functional of the ground state is supported on a sphere  $S_{|\mathbf{q}_0|}$  whose radius is  $\mathbf{q}_0$ . In particle physics we have become rather sophisticated, and when one writes down the Landau-Ginsburg action, one usually requires that the expansion in the order parameter to be under control. For example, this means that there is a limit in which this expansion becomes exact. In the case at hand, this is not the situation, and it is actually very complicated, nevertheless one follows the usual Landau-Ginsburg expansion.

The term which follows the quadratic interaction is a cubic term

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3, \quad (10)$$

$$\mathcal{L}_3 = \int d^3\mathbf{q}_1 d^3\mathbf{q}_2 d^3\mathbf{q}_3 \varrho(\mathbf{q}_1) \varrho(\mathbf{q}_2) \varrho(\mathbf{q}_3) \delta(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) \times \\ B(|\mathbf{q}_1|^2, |\mathbf{q}_2|^2, |\mathbf{q}_3|^2, \mathbf{q}_1 \cdot \mathbf{q}_2, \mathbf{q}_1 \cdot \mathbf{q}_3, \mathbf{q}_2 \cdot \mathbf{q}_3) \quad (11)$$

I am going to assume for the purpose of illustration, as Landau did, that this is a good perturbation, namely that when one considers  $\mathcal{L}_3$  one is going already to assume that the support of  $\varrho$  comes from only those values of  $\mathbf{q}$  such that  $|\mathbf{q}_1|^2 \cong |\mathbf{q}_2|^2 \cong |\mathbf{q}_3|^2 \cong |\mathbf{q}_0|^2$ . This was determined by  $\mathcal{L}_2$ .

In (11), once again, the delta function  $\delta(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3)$  enforces the explicit translational invariance, and the dependence of  $B$  on the momentum respects both translational and rotational invariance.

The integral in the  $\mathbf{q}$ 's is not over all possible values, but only over those whose lengths is determined by  $|\mathbf{q}_0|^2$ , which in turn was fixed by  $\mathcal{L}_2$ .

An additional structure emerges due to the effect of the delta function  $\delta(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3)$ . It restricts the candidates for the ground state, to have support on at least three different values for the  $\mathbf{q}_i$ . The three vectors appearing need to sum up to give a triangle, see fig 7.

Actually they are six if the field is real since one needs

$$\varrho(\mathbf{q}) = \varrho(-\mathbf{q}). \quad (12)$$

Thus one has at least six components of  $\varrho(\mathbf{q})$  which do not vanish. In general, instead of  $\varrho(\mathbf{q})$  having support on all values of a sphere, they are now broken into triplets where the  $\mathbf{q}_i$  have to sum together to form triangles, (fig. 7). In this manner also rotational invariance is spontaneously broken.

Let's be even more explicit, because we have used the approximation that all the  $\mathbf{q}_i$  have the same length, the  $\mathbf{q}_i$  that tessellate the sphere, have to form equilateral triangles, as in fig. 7. Equilateral triangles single out a specific angle  $60^\circ$ , that is a spontaneous breaking of rotational invariance. One has obtained a non zero value for  $\mathbf{q}$ , and one has derived that the ground state is built out of objects which have to sum up to form triangles which are equilateral and thus have a  $60^\circ$  angle.

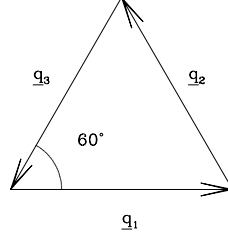
From energetic and combinatorial considerations, one finds that to be on an extremum, one needs all the values of  $\varrho(\mathbf{q}_i)$  to be equal

$$|\varrho(\mathbf{q}_i)|^2 = |\varrho(\mathbf{q}_0)|^2, \quad (13)$$

which leads to

$$|\varrho(\mathbf{x})|^2 = n |\varrho(\mathbf{q}_0)|^2. \quad (14)$$

There are a couple of general way to distribute the triplets, one in which each  $\mathbf{q}_i$  appears in only one of the triplets, and another in which each value of  $\mathbf{q}_i$  does participate in two triplets. The number of elements is proportional to  $n$  in both cases, being either  $2n/3$  or  $4n/3$ . When one does the analysis, and one estimates the value of  $\mathcal{L}_3$ , one finds that it decreases as the inverse of  $\sqrt{n}$



**Fig. 7.** The sphere  $\mathcal{S}_{|q_0|}$  is triangulated due to the presence of a cubic term in the Lagrangian. Since in this approximation all the sides of the triangles have the same length, their angles are determined to be  $60^\circ$ . Rotational invariance is thus spontaneously broken.

$$\mathcal{L}_3 \sim \frac{|\varrho(q_0)|^2}{\sqrt{n}}. \quad (15)$$

Thus the ground state will be obtained for some finite value of  $n$ . One needs to consider only a finite number of triplet configurations when one searches for the extrema of the free energy. Just three, i.e six participants lead to the following density distribution

$$\varrho(x, y) = \pm \left(\frac{2}{3}\right)^{1/2} \varrho_{q_0} \left[ \cos(q_0 x) + 2 \cos\left(\frac{1}{2}q_0 x\right) \cos\left(\frac{\sqrt{3}}{2}q_0 y\right) \right]. \quad (16)$$

The corresponding free energy is

$$\mathcal{L}_3^{n=3} = \frac{2B\varrho_{q_0}^3}{3\sqrt{3}}. \quad (17)$$

For the case of two spatial dimensions it turns out that if  $\varrho(q_0) > 0$  it is advantageous to form a triangular lattice, while if  $\varrho(q_0) < 0$ , the dual lattice, which is a honeycomb lattice, is formed.

This required only studying the minimal possible configuration. In three spatial dimensions, this would be a candidate for a two dimensional lattice in three dimensions, if one wishes some type of compactification.

In three dimensions one needs to consider also larger configurations to obtain the extrema. The next candidate configuration has six ( $n = 6$ ), i.e twelve values of  $\mathbf{q}$ . This is a more complicated configuration, whose density distribution is

$$\varrho(x, y, z) = \frac{2}{\sqrt{3}} \varrho_{q_0} \left[ \cos\left(\frac{\sqrt{2}}{2} q_0 x\right) \cos\left(\frac{\sqrt{2}}{2} q_0 y\right) + \cos\left(\frac{\sqrt{2}}{2} q_0 x\right) \cos\left(\frac{\sqrt{2}}{2} q_0 z\right) + \cos\left(\frac{\sqrt{2}}{2} q_0 y\right) \cos\left(\frac{\sqrt{2}}{2} q_0 z\right) \right]. \quad (18)$$

This actually describes a BCC lattice (in real space). The value of  $\mathcal{L}_3$  is larger than that one of the former configuration

$$\mathcal{L}_3^{n=6} = \frac{4B\varrho_{q_0}^3}{3\sqrt{6}} > \mathcal{L}_3^{n=3}, \quad (19)$$

and leads to the extrema of the free energy, being the most stable configuration.

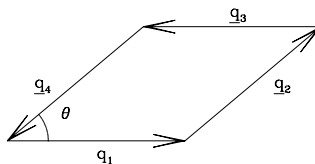
From amazingly simple considerations, one has a prediction that solids in three dimensions are all BCC lattices, a very universal description of the system. Before confronting this claim with the data one needs to recall that the transitions between solids and liquids are not second order transitions, they are actually first order transitions. So one may question the validity of universality claims in this context. However, it turns out that in many cases one can arrange that the solidifications occur as a weak first-order transitions, in which case approximate universality properties can be present.

Returning to the data and following [6], one discovers that about 40 metals, which are on the left of the periodic table, (excluding Magnesium (Mg)), form near the solidification point a BCC configuration.

I will repeat the difficulties of the analysis and the argumentations to proceed with it. The transition is first order - the fact that in many cases it is a weak first order transition softens this problem. There is no true expansion parameter in the problem. The microscopic theory constructing  $A$  and  $B$  is very phenomenological, and therefore the real relative stability of the metal is a very delicate matter. Even taking all this into account the result and its agreement with a large body of the experimental data is striking.

Consider what would have happened without a cubic term. In that case, the term following the quadratic term would be  $\mathcal{L}_4$ , which schematically would assume the form

$$\mathcal{L}_4 = \int d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3 d\mathbf{q}_4 \delta(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4) \times C(|\mathbf{q}_1|^2, |\mathbf{q}_2|^2, |\mathbf{q}_3|^2, |\mathbf{q}_4|^2, \mathbf{q}_1 \cdot \mathbf{q}_2, \mathbf{q}_1 \cdot \mathbf{q}_3, \dots), \quad (20)$$



**Fig. 8.** In the absence of a cubic term, a quartic term would not suffice classically to induce a spontaneous breaking of rotational invariance. A rhombus does not single out a preferred angle  $\theta$

where the delta function enforces translational invariance, and the function  $C$  should be built by such invariants that maintain both rotational and translational invariance.

This does not break rotational invariance, because unlike the case of triangles, the configurations which are enforced now, assuming perturbation theory, are those of quadrilaterals with equal sides. But for a rhombus (fig. 8) no preferred angle is singled out. The rotational invariance is not broken. Fortunately there is no microscopic symmetry consideration that rules out the cubic term.

Another interesting type of lattices are the Abrikosov lattices formed of vortices, which we do not discuss here.

## 2.1 String Theory Compactifications

What has been described above has a very solid basis in nature. What we will describe next is of a much more speculative nature, and it is based on work by Elitzur, Forge and myself [4], in which we try to address the issues of compactification in String Theory. There are several attitudes one might adopt regarding compactification. One, which makes a lot of sense, is to say that the Universe starts up very small, and the issue of compactification is an issue of explaining why four dimensions became very large, while the rest of the dimensions remain small. This is not what I am going to discuss here.

Here, I discuss possible dynamical aspects of compactification taking in account some of the hints learnt from the case of solid state physics. I don't have much confidence in human imagination when it is totally detached from reality, and I would hope that many of the hints available in nature will be useful to understand other phenomena. In particle physics one has learned quite a lot from the dynamics of solid state physics, and statistical mechanics systems.

Returning to the case at hand, we have just reviewed a system which has lost most of its rotational and translational invariance, and we want to see how a similar thing might happen in String Theory. One of the key ingredients driving this behavior is the presence of a bulk tachyon.

There are actually at least three types of tachyons/instabilities with whom one is familiar right now in String theory. One is the Bosonic closed String Theory tachyon. This instability could well be an incurable one, nevertheless let's try and follow it.

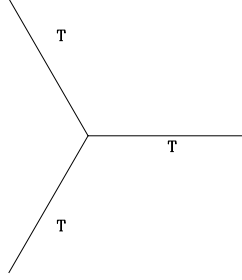
The other types of instabilities, which we will discuss later, are an instability in Open String Theory, an open string tachyon, and also localized bulk tachyons.

For the moment we focus on bulk tachyons, which will be one key ingredient. Due to them it is preferable for a system in String Theory containing a tachyon to have a support on a non-zero value of  $q^2$ . One can see this from the form of the tachyon, whose vertex operator is the following

$$T(x) = e^{iq_0 x} + h.c. \quad (21)$$

To obtain a dimension  $(1, 1)$  operator, one needs  $q_0 \neq 0$ . Tachyons do give us the starting point that appears in Landau theory of solidification (note that here it is not a minimum consideration). The second key ingredient that we need for Landau's theory of solidification, in order to obtain not only the breakdown of translational invariance, but also of rotational invariance, is the presence of a cubic term. We know from the OPE (operator-product-expansion) that three tachyons do couple together, (see fig.9). In particular, the OPE between two tachyons does contain a third tachyon. So we have in a such a theory a  $T^3$  term. One indeed has the necessary ingredients to try to follow if tachyons could lead to the spontaneous breaking of rotational and translational invariance in String Theory, and maybe also to compactification.

In order to be more concrete, we followed the ideas of [7], [8] and tried to handle in a reliable fashion almost marginal operators. Consider a tachyon which is not an exact  $(1, 1)$  operator, but one which has  $q_0^2 = 2 - \epsilon$ . We will also look at the subset of the full string background, a subset which contains a  $c = 2$  sector. We will not deal here with the question of how the total central charge remains at the appropriate value, which is zero and how to dress operators.

**Fig. 9.** Tachyonic cubic vertex

As an illustration, consider the subset of the backgrounds which are string moving in flat space, where the piece of the lagrangian on which we focus is

$$\mathcal{L} = \partial X^1 \bar{\partial} X^1 + \partial X^2 \bar{\partial} X^2 + T(X^1, X^2). \quad (22)$$

From Landau's theory of solidification, we know that because the system has support on a  $\mathbf{q}_0 \neq 0$ , and because the free energy of the system contains a cubic coupling, we can try to build the triplet, which again are actually six vectors, so they get a support in an appropriate way, i.e such that they break translational and rotational invariance.

The  $60^\circ$  angle, discussed in the solidification case, manifests itself in a suggested tachyon configuration:

$$T(X^1, X^2) = \sum_{a=1}^3 T_a \cos\left(\sum_i^2 k_i^a X^i\right), \quad (23)$$

where the three momenta  $\mathbf{k}^1$ ,  $\mathbf{k}^2$ , and  $\mathbf{k}^3$  are the following

$$\mathbf{k}^1 = k(1, 0) \quad \mathbf{k}^2 = k(-1/2, \sqrt{3}/2) \quad \mathbf{k}^3 = k(-1/2, -\sqrt{3}/2). \quad (24)$$

All of them have  $\mathbf{k}^2 = 2 - \varepsilon$ , and the structure is very similar to that of the  $SU(3)$  root lattice (see fig. 10), as before for a every  $k_i$ , there is also the corresponding  $-k_i$  contribution.

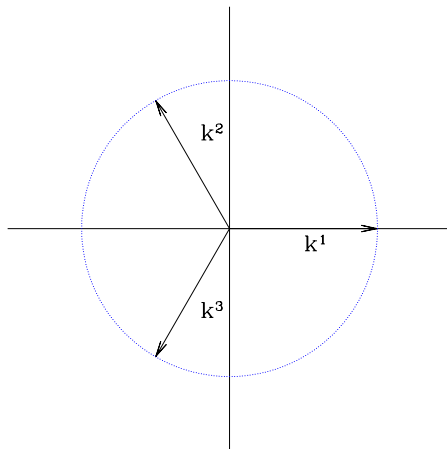
One can simplify the tachyon potential by taking the ansatz for the amplitudes  $T_a = T$ .

The Lagrangian one needs to solve is the following

$$\mathcal{L} = \partial X^1 \bar{\partial} X^1 + \partial X^2 \bar{\partial} X^2 + T(X^1, X^2), \quad (25)$$

and actually one can show that the beta function of the tachyon alone vanishes to order  $\varepsilon$ . So (23) is a solution of the approximate tachyon equations of motion. This means that had it been up to the tachyon alone one





**Fig. 10.** SU(2) roots

would have obtained the lattice, perhaps some honeycomb or triangular lattice, which would break both translational and rotational invariance. However, this system contains also gravity so one needs to see what is the influence of the formation of such a lattice on gravity. As shown in [4], the beta function for the graviton  $\beta_{G_{\mu\nu}}$  vanishes (at leading order in  $\alpha'$ ) if

$$\beta_{G_{\mu\nu}} = -R_{\mu\nu} + \nabla_\mu T \nabla_\nu T = -R_{\mu\nu} + \frac{3}{2} \varepsilon^2 \delta_{\mu\nu} = 0. \quad (26)$$

For  $D = 2$ , due to the Liouville theorem  $R_{\mu\nu}$  can be written as  $R_{\mu\nu} = a \delta_{\mu\nu}$ , so one can solve the equation by forming a two-dimensional sphere.

This is actually a highlight of a model for compactification. We started by having just a tachyon. The tachyon would have produced the lattice on its own, but because of the presence of the gravity the lattice of tachyons actually causes the compactification of space to a sphere.

However, it turns out, and details are presented in [4], that unfortunately this result is not obtained in a desired reliable approximation. The main problem is that in order to do reliable perturbation theory, we need to do a plane waves expansion, with the wave lengths representing a nearly marginal operator. However, the moment the sphere is formed the topology changes, and the change of topology means that one should now expand the fields in terms of spherical harmonics  $Y_{l,m}$ . This topological obstruction takes away the reliability of our calculation. Some defects may form in order to resolve this topological problem, and one conjecture we had at that time was that ac-

tually parafermions, which are defects, form to resolve the tension. A more complex form of compactification emerges.

Once again, recall that actually the system, when fully considered, has to be coupled to the dilaton, in order to maintain the total central charge. According to the Zamolodchikov theorem [7], once the system starts to flow, the central charge decreases from 2 and this on its own breaks the balance. In a sense, in the case of bulk tachyons we were tantalizingly close to obtain an explicit dynamical mechanism for compactification. However, due to topological obstructions, what was a solution for the beta function locally in space, it cannot be a global solution without taking into account other effects. We will return to the breaking of translational invariance in the different context of the open string tachyon.

## 2.2 Liquid Crystals

The tachyon is a scalar order parameter, String Theory has additional fields which carry indexes. In particular, one might think that if one looks for a similarity to our universe, maybe one should consider the phase of liquid crystals. Such systems are translational invariant in some directions but not in other (see fig. 11). We will give now examples of that.

There are various types of liquid crystals and one can ask what is the Landau-Ginsburg theory of them. Actually, one can also ask about vector potential systems which are described, as gauge fields are, by vector order parameters. Such systems include detergents which possess a hydrophobic and a hydrophilic pole, and play a crucial role in cleaning our garments. One can try to extract from  $\mathbf{p}(\mathbf{r})$  the various invariants one wants to use in order to describe this system, such as  $\text{div}\mathbf{p}$ ,  $\text{curl}\mathbf{p}$ ,  $s_{\alpha\beta} = \partial_\alpha p_\beta + \partial_\beta p_\alpha$ .

It turns out that one can write down a Landau-Ginsburg theory for detergents, which explains many of their very fascinating properties.

Consider the case of liquid crystals, these can also be written by choosing as an order parameter particular spherical harmonic functions.

For illustrative purposes, we give the dependence of the density  $\phi$  on the angles and on the coordinates<sup>2</sup>

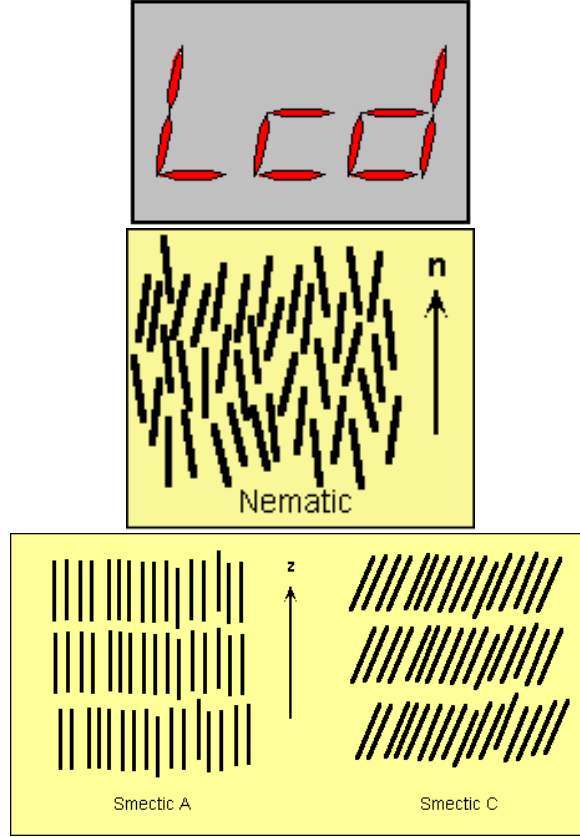
$$\phi = \sum_i \mu_i Y_2^2(\theta_i, \phi_i) e^{i\mathbf{k}_i \cdot \mathbf{r}_i} + h.c. \quad (27)$$

By assuming the ansatz  $\mu_i = \mu$ , the effective Landau-Ginsburg free energy is given below

$$F \sim (\alpha_0 + dk + ck^2)\mu^2 - \beta\mu^3 + r\mu^4, \quad (28)$$

---

<sup>2</sup> The index structure of  $\phi$  has been omitted.



**Fig. 11.** Various phases of liquid crystals breaking. These systems exhibit asymmetrical breaking of translational and rotational invariance.

from which one can extract the properties of nematic, smectic A and smectic C properties, and many other exciting things for which we refer to the literature [6].

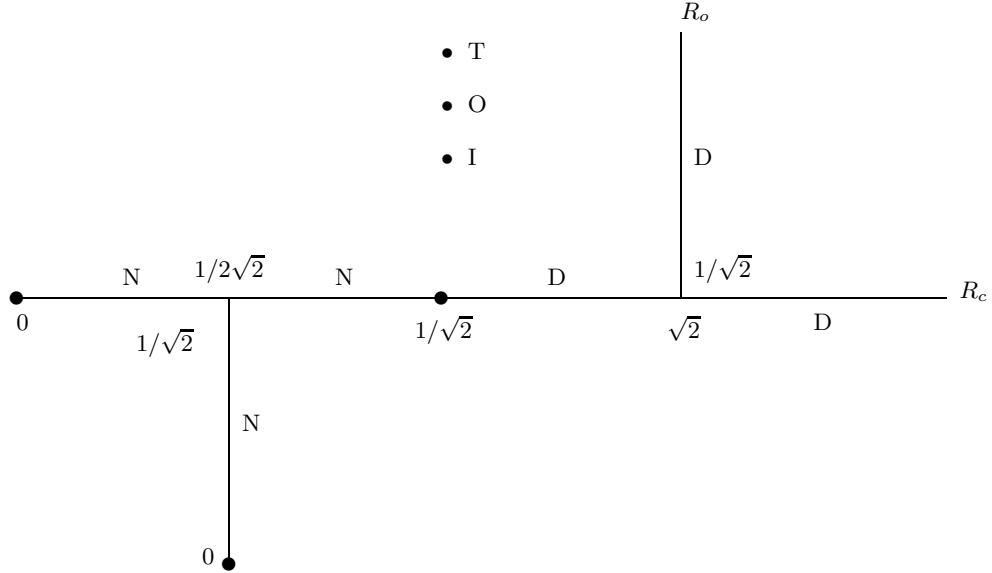
### 2.3 Boundary Perturbations

Next I discuss an example where a breakdown of spatial translational symmetry actually clearly occurs. As mentioned above one can formulate an intuitive theorem in the bulk, which states that under the renormalization group flow, the value of the Virasoro central charge  $c$  decreases from its UV value to a smaller IR value. This is due to the integrating out of the degrees of freedom and applies to the unitary sectors of String Theory. In String Theory with its ghosts the total central charge vanishes. One can imagine mechanisms by which the central charge of the ghosts increases [9], but basically one needs

to couple the two systems maintaining a total vanishing central charge. This can be done for example with the help of a linear dilaton and leads to very interesting questions and results. Generically, the matter central charge will decrease to zero leaving one with just a  $c = 0$  topological theory, but there are also other possibilities. The central charge is related to the anomaly which exists in the bulk. On the other hand, when one considers the boundary theory, there are no gravitational anomalies in it. Thus in that case one can consider tachyonic open string theory perturbations. In the example given in the action below

$$S = \int_{\Sigma} \mathcal{L}_{CFT} + \int_{\partial\Sigma} g \mathcal{O}_{Rel.}, \quad (29)$$

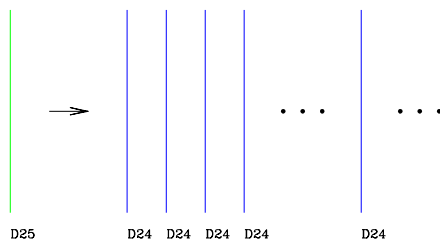
the bulk theory is defined on the surface  $\Sigma$  and on its boundary  $\partial\Sigma$  one adds a relevant operator  $\mathcal{O}_{Rel.}$ . There is a boundary renormalization group flow, which does not change the bulk central charge and therefore does not lead to all the problems associated with tachyons in the bulk.



**Fig. 12.** Map of the preferred boundary conditions in the  $c = 1$  moduli space, N stands for Neuman and D for Dirichlet boundary conditions [9]

One can associate a term in the boundary which measures the effective number of degrees of freedom, this has been done by various authors [10], [12],[11].

One can prove moreover that one can define such a function whose value also decreases when the theory flows on the boundary. All this while not requiring an adjustment of the total central charge. What happens for example is that the theory flows from Neuman(N) to Dirichlet(D) boundary conditions [13], or in other words the branes may dissolve or may be created under such a flow. In the figure below (fig.12) we give an example of a very simple compactification for which one can identify what are the stable configurations, describing when the system chooses to obey Dirichlet and when the system chooses to obey Neuman boundary conditions [9].



**Fig. 13.** A lattice of D24 branes is formed from a D25 brane in the presence of a boundary tachyon

This can be used even further if one changes the relevant operator added on the boundary into a Sine-Gordon one. In that case one can actually have situations where one breaks translational invariance in space-time by a  $D - 25$  brane for example dissolving into a lattice of  $D - 24$  branes [14], (fig.13). Again such a situation will lead also to a reduction of the original amount of supersymmetry. Thus the idea of spontaneous breaking of spatial translational invariance is demonstratively realized in String Theory by the presence of open string tachyons.

### 3 Spontaneous Breaking of Time Translational Invariance and of Supersymmetry

Next I will discuss a somewhat different mechanism which may allow the possibility of a spontaneous breaking of time-translational invariance. For that it is useful to consider conformal and superconformal quantum mechanics. One way to motivate the interest in such systems is to recall some basic facts concerning the validity of a perturbative expansion.

Consider the Hamiltonian,

$$H = \frac{p_q^2}{2m} + \frac{1}{2}gq^n . \quad (30)$$

One may wonder if it is possible to make a meaningful perturbative expansion in terms of small or large  $g$  or small or large  $m$ . To answer this question one needs to find out if one can remove the  $g, m$  dependence from the operators, and relegate it to the total energy scale. This type of rescaling is used for discussing the harmonic oscillator. One attempts to define a new set of dimensionless canonical variables  $p_x, x$  that preserve the commutation relations

$$[p_q, q] = [p_x, x]\hbar , \quad (31)$$

and

$$H = h(m, g)\frac{1}{2}(p_x^2 + x^n) . \quad (32)$$

The following decomposition:

$$q = f(m, g)x , \quad p_q = \frac{1}{f(m, g)}p_x , \quad (33)$$

gives

$$2H = \frac{p_x^2}{mf^2(m, g)} + gf(m, g)^n x^n , \quad (34)$$

and so one may choose

$$gf(m, g) = \left( \frac{1}{mf(m, g)^2} \right)^{\frac{1}{n+2}} . \quad (35)$$

The Hamiltonian becomes:

$$H = g^{1-\frac{n}{n+2}} m^{-\frac{n}{n+2}} \frac{1}{2}(p_q^2 + q^n) . \quad (36)$$

The role of  $g$  and  $m$  is indeed just to determine the overall energy scale. They may not serve as meaningful perturbation parameters.

This does not apply to the special case of  $n = -2$ , the case of conformal quantum mechanics, where  $g$  can be a real perturbative parameter.

### 3.1 Conformal Quantum Mechanics: A Stable System With No Ground State Breaks Time Translational Invariance

Consider the Hamiltonian

$$H = \frac{1}{2}(p^2 + gx^{-2}) \quad (37)$$

for a positive value of  $g$  [15].

$H$  is part of the following algebra:

$$[H, D] = iH, \quad [K, D] = iK, \quad [H, K] = 2iD. \quad (38)$$

It is an  $SO(2,1)$  algebra, one representation of which is:

$$D = -\frac{1}{4}(xp + px), \quad K = \frac{1}{2}x^2 \quad (39)$$

with  $H$  is given above. The Casimir is given by:

$$\frac{1}{2}(HK + KH) - D^2 = \frac{g}{4} - \frac{3}{16}. \quad (40)$$

In the Lagrangian formalism the system is described by:

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 - \frac{g}{x^2}), \quad S = \int dt \mathcal{L}. \quad (41)$$

Symmetries of the action  $S$ , and not of the Lagrangian  $\mathcal{L}$  alone, are given by:

$$t' = \frac{at + b}{ct + d}, \quad x'(t') = \frac{1}{ct + d}x(t), \quad (42)$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det A = ad - bc = 1 \quad (43)$$

$H$  acts as translation

$$A_T = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}, \quad t' = t + \delta. \quad (44)$$

$D$  acts as dilation

$$A_D = \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}, \quad t' = \alpha^2 t. \quad (45)$$

$K$  acts as a special conformal transformation

$$A_K = \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}, \quad t' = \frac{t}{\delta t + 1}. \quad (46)$$

The spectrum of the Hamiltonian (37) is the open set  $(0, \infty)$ , the spectrum is therefore continuous and bounded from below. The wave functions are given by:

$$\psi_E(x) = \sqrt{x} J_{\sqrt{g+\frac{1}{4}}}(\sqrt{2E}x) \quad E \neq 0. \quad (47)$$

The zero energy state is given by  $\phi(x) = x^\alpha$ :

$$H = \left( -\frac{d^2}{dx^2} + \frac{g}{x^2} \right) x^\alpha = 0. \quad (48)$$

This implies

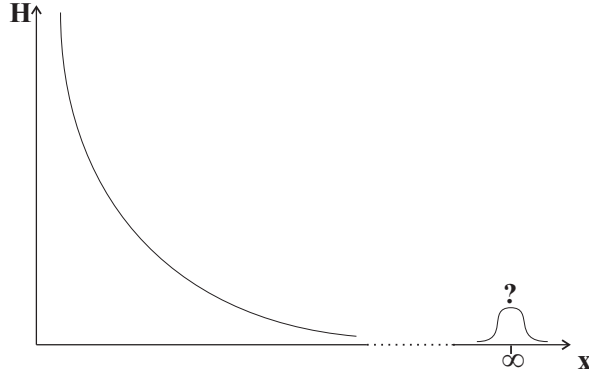
$$g = -\alpha(\alpha - 1), \quad (49)$$

solving this equation gives

$$\alpha = -\frac{1}{2} \pm \frac{\sqrt{1+4g}}{2}. \quad (50)$$

This gives rise to two independent solutions and by completeness these are all the solutions. The case  $\alpha_+ > 0$ , does not lead to a normalizable solution since the function diverges at infinity.  $\alpha_- < 0$ , is not normalizable either since the function diverges at the origin (a result of the scale symmetry).

Thus, there is no normalizable (not even plane wave normalizable)  $E=0$  solution!



**Fig. 14.** Their no normalisable ground state for this potential



Most of the analysis in field theory proceeds by identifying a ground state and the fluctuations around it. How do we deal with a system in the absence of a ground state?

One possibility is to accept this as a fact of life. Perhaps it is possible to view this as similar to cosmological models that also lack a ground state, such those with Quintessence. In field theory such systems have no finite energy states in the spectrum at all. Only time dependent states are allowed. In the presence of an appropriate cutoff and in quantum mechanics it is only the potential lowest energy state which is disallowed.

Another possibility is to define a new evolution operator that does have a ground state

$$G = uH + vD + wK . \quad (51)$$

This operator has a ground state if  $v^2 - 4uw < 0$ . Any choice explicitly breaks scale invariance. Take for example

$$G = \frac{1}{2} \left( \frac{1}{a} K + aH \right) \equiv R , \quad (52)$$

$a$  has the dimension of a length. The eigenvalues of  $R$  are

$$r_n = r_0 + n , \quad r_0 = \frac{1}{2} \left( 1 + \sqrt{g + \frac{1}{4}} \right) . \quad (53)$$

This is a breaking of scale invariance by a dictum and not by the dynamics of the system. Nevertheless it is very interesting to search for a physical interpretation of this. Surprisingly this question arises in the context of black hole physics [17]. Consider a particle of mass  $m$  and charge  $q$  falling into a charged black hole. The black hole is BPS, meaning that its mass  $M$  and charge  $Q$  are related, in the appropriate unites, by  $M = Q$ .

The blackhole metric and vector potential are given by:

$$ds^2 = - \left( 1 + \frac{M}{r} \right)^{-2} dt^2 + \left( 1 + \frac{M}{r} \right)^2 (dr^2 + r^2 d\Omega^2) , \quad A_t = \frac{r}{M} \quad (54)$$

Now consider the near Horizon limit, i.e.  $r \ll M$ , which we will reach by taking  $M \rightarrow \infty$  and keeping  $r$  fixed. This produces an  $AdS_2 \times S^2$  geometry

$$ds^2 = - \left( \frac{r}{M} \right)^2 dt^2 + \left( \frac{M}{r} \right)^2 dr^2 + M^2 d\Omega^2 . \quad (55)$$

We also wish to keep  $M^2(m - q)$  fixed as we scale  $M$ . This means we must scale  $(m - q) \rightarrow 0$ , that is the particle itself becomes BPS in the limit.

The Hamiltonian for this falling in particle in this limit is given by our old friend:

$$H = \frac{p_r^2}{2m} + \frac{g}{2r^2}, \quad g = 8M^2(m - q) + \frac{4l(l+1)}{M}. \quad (56)$$

For  $l = 0$ , we have  $g > 0$  and there is no ground state. This is associated with the coordinate singularity at the Horizon. The change in evolution operator is now associated with a change of time coordinate. One for which the world line of a static particle passes through the black hole horizon instead of remaining in the exterior of the space time. In any case, the consequence of removing the potential lowest energy state of the system from the spectrum can be described as a breaking of time translational invariance.

### 3.2 Superconformal Quantum Mechanics: A Stable System With No Ground State Breaks Also Supersymmetry

The bosonic conformal mechanical system had no ground state. The absence of a  $E = 0$  ground state in the supersymmetric context leads to the breaking of supersymmetry. This breaking has a different flavor from that which was discussed for the spatial translations. We next examine the supersymmetric version of conformal quantum mechanics [1], [16], to see if indeed supersymmetry is broken. The superpotential is chosen to be

$$W(x) = \frac{1}{2}g \log x^2, \quad (57)$$

yielding the Hamiltonian:

$$H = \frac{1}{2} \left[ \left( p^2 + \left( \frac{dw}{dx} \right)^2 \right) 1 - \frac{d^2W}{dx^2} \sigma_3 \right]. \quad (58)$$

Representing  $\psi$  by  $\frac{1}{2}\sigma_-$  and  $\psi^*$  by  $\frac{1}{2}\sigma_+$  gives the supercharges:

$$Q = \psi^+ \left( -ip + \frac{dW}{dx} \right), \quad Q^+ = \psi \left( ip + \frac{dW}{dx} \right). \quad (59)$$

One now has a larger algebra, the superconformal algebra,

$$\begin{aligned} \{Q, Q^+\} &= 2H, \quad \{Q, S^+\} = g - B + 2iD, \\ \{S, S^+\} &= 2K, \quad \{Q^+, S\} = g - B - 2iD. \end{aligned} \quad (60)$$

A realization is:

$$B = \sigma_3, \quad S = \psi^+ x, \quad S^+ = \psi x. \quad (61)$$

The zero energy solutions are

$$\exp(\pm W(x)) = x^{\pm g}, \quad (62)$$

neither solution is normalizable.

$H$  factorizes:

$$2H = \begin{pmatrix} p^2 + \frac{g(g+1)}{x^2} & 0 \\ 0 & p^2 + \frac{g(g-1)}{x^2} \end{pmatrix}, \quad (63)$$

and we may solve for the full spectrum:

$$\psi_E(x) = x^{1/2} J_{\sqrt{\mathbf{v}}}(x\sqrt{2E}), \quad E \neq 0, \quad (64)$$

where  $\mathbf{v} = g(g-1) + 1/4$  for  $N_F = 0$  and  $\mathbf{v} = g(g+1) + 1/4$  for  $N_F = 1$ .

The spectrum is continuous and there is no normalizable zero energy state. One must interpret the absence of a normalizable ground state. It is also possible to define a new operator which has a normalizable ground state. By inspection the operator (52) can be used, provided one makes the following identifications:

$$\begin{aligned} N_F = 1 & \quad g_B = g_{susy}(g_{susy} + 1), \\ N_F = 0 & \quad g_B = g_{susy}(g_{susy} - 1). \end{aligned} \quad (65)$$

Thus the spectrum differs between the  $N_F = 1$  and  $N_F = 0$  sectors and supersymmetry would be broken. One needs to define a whole new set of operators:

$$\begin{aligned} M &= Q - S & M^+ &= Q^+ - S^+ \\ N &= Q^+ + S^+ & N^+ &= Q + S^+ \end{aligned} \quad (66)$$

which produces the algebra:

$$\begin{aligned} \frac{1}{4}\{M, M^+\} &= R + \frac{1}{2}B - \frac{1}{2}g \equiv T_1, \\ \frac{1}{4}\{N, N^+\} &= R + \frac{1}{2}B + \frac{1}{2}g \equiv T_2, \\ \frac{1}{4}\{M, N\} &= L_- & \frac{1}{4}\{M^+, N^+\} &= L_+, \\ L_{\pm} &= -\frac{1}{2}(H - K \mp 2iD) \end{aligned} \quad (67)$$

$T_1, T_2, H$  have a doublet spectra. “Ground states” are given by:

$$T_1|0\rangle = 0; \quad T_2|0\rangle = 0; \quad H|0\rangle = 0. \quad (68)$$

In this setup one can also exhibit [1] how the in the presence of a breaking of a space time symmetry, global  $\mathcal{N} = 2$  can be broken only to  $\mathcal{N} = 1$ . A physical context arises when one considers a supersymmetric particle falling

into a black hole [17], [18]. This is the supersymmetric analogue of the situation already discussed.

One should mention again that there is a dictum in the way one has broken scale/conformal invariance in the problem. It is amusing to mention that if one takes the dictated ground state and decomposes it in terms of the energy eigenstates, then one usually gets that the new ground state looks like a thermal distribution of the old ground states. This looks very attractive and it is related to black holes, which as mentioned above do come up.

Another example where such breakdown of time-translational invariance may occur is the Liouville model. Also there, there is no normalizable ground state. For works on the possible breakdown of translational invariance in the two dimensional Liouville model, see [19],[20].

Beyond  $d = 2$ , we can also mention that in four dimensions in  $\mathcal{N} = 1$  supersymmetric theories, where the number of flavors  $N_F$  is smaller than the number of colors  $0 < N_F < N_C$ , one gets [22],[21]. This is another situation where the spectrum is bounded from below but there is no ground state (fig.15). The spectrum is open, and actually in the presence of a cutoff such systems have no finite energy states at all, which is a very interesting as far as Cosmology is concerned.

## 4 Spontaneous Breaking of Conformal Invariance

Fubini also suggested to discuss such situations in a general number of dimensions [23]. He researched it in a scientific environment which did not yet fully realize that interacting finite theories might exist in various number of dimensions. Therefore much of his analysis was of a classical nature. He emphasized the conformal features of the system, and we are going to discuss the breakdown of conformal invariance. The discussion of the breakdown of time translational invariance brought us to conformal theories and now we are discussing also the breakdown of the conformal invariance.

If one considers a theory with only one scalar field, a general classic conformal invariant is given by the following Lagrangian

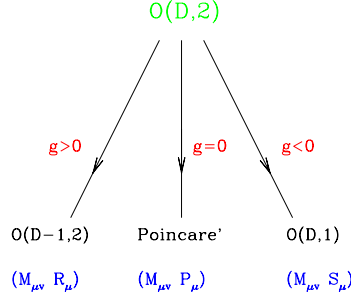
$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - g \phi^{\frac{2d}{d-2}}. \quad (69)$$

The symmetry of the system is the bosonic,  $O(d, 2)$  symmetry, and the generator are  $M_{\mu\nu}$ ,  $P_\mu$ , of the Poincaré group, the special conformal transformation generator  $K_\mu$ , and the dilatation  $D$ . The dictum of Fubini in this case is that the ground state is not translational invariant, this is not accompanied by any dynamical calculation. The vacuum expectation value  $\langle \phi(x) \rangle$ , is  $x$  dependent, and actually it looks very much like an instanton

$$\langle \phi(x) \rangle = b \left( \frac{a^2 + x^2}{2a} \right)^{-\frac{d-2}{2}}, \quad (70)$$

which is a solution of the equation of motion

$$\partial^2 \phi(x) - 2g \frac{d}{d-2} \phi^{\frac{d+2}{d-2}}(x) = 0 . \quad (71)$$



**Fig. 15.** The sign of the quartic coupling  $g$  determines the symmetry breaking patterns of the symmetry group  $O(d, 2)$

By choosing this to be the vacuum, (again I emphasize, this is by dictum), one breaks down the  $O(d, 2)$  symmetry, (as in fig. 15) in the following fashion: if the coupling  $g$  of the scalar self-interaction is positive the theory breaks down to  $O(d - 1, 2)$  and the resulting symmetries are  $M_{\mu\nu}$ ,  $R_{\mu}$ . If  $g < 0$  the symmetry breaks to  $O(d - 1)$ , generated by  $M_{\mu\nu}$ ,  $S_{\mu}$ , where

$$S_{\mu} = \frac{1}{2} \left( a P_{\mu} - \frac{1}{a} K_{\mu} \right) . \quad (72)$$

If  $g = 0$  one remains with Poincaré invariance, (fig. 15). In the de Sitter example, which occurs for  $g > 0$ , one can show again that there are signatures of temperature. A question which at the time seemed interesting was: Does a spontaneous breaking of conformal invariance require also the breakdown of translational invariance? Examples were since found where this is not the case. Counter examples to the idea that the breaking of conformal invariance must drive a breaking of Supersymmetry were discovered. We will discuss in more detail some such examples. One can break scale invariance without breaking rotational or translational invariance. We also mention briefly that conformal invariance and scale invariance are not always equivalent, and in a

set of works, (see for example [24]), one can show that scale invariance leads under some certain conditions to conformal invariance.

That is the case when the spectrum of the theory is discrete, such as for a two dimensional sigma model description in which the target space is compact. But for non-compact target spaces one can find counter examples [25], in which scale invariance does not lead to conformal invariance. In recent years it has been fully realized that theory which are quantum mechanically scalar invariant and finite may exist in  $d = 2, 3, 4, 5, 6$  dimensions. Such theories can exhibit spontaneous breaking, for example the  $d = 4$ ,  $\mathcal{N} = 4$  Super Yang-Mills with  $SU(N)$  gauge group is characterized by the following spectrum

$$(A_\mu^a, \lambda^a, \phi^a + i\varrho^a).$$

The theory is parameterized by the complex parameter  $ig + \theta$ , where  $g$  is the coupling constant and  $\theta$  is the  $\theta$  angle. Such a theory has flat directions which allow phases where either  $\langle \phi \rangle$  vanishes and the theory is realized in a conformal manner, or a phase in which  $\langle \phi \rangle \neq 0$  along flat directions. This is the Coulomb phase, in which the gauge group  $SU(N)$  may be reduced all the way to  $U(1)^N$ , where  $N$  is the rank of the gauge group. This is the maximum possible breaking of the gauge group when the fields are in the adjoint representation. In such a case, scale invariance is broken spontaneously and the vacuum energy remains zero, and there is no breakdown of either translational invariance or Supersymmetry. Such a theory will have a Goldstone boson, associated with the spontaneous breaking of scale invariance, which is called the dilaton. This is a *true* dilaton worthy of his name. It is interesting to note that in such a system the vacuum energy is not influenced by the value of  $\langle \phi \rangle$ , and it vanishes in all the phases.

## 5 $O(N)$ Vector Models in $d = 3$ : Spontaneous Breaking of Scale Invariance and The Vacuum Energy

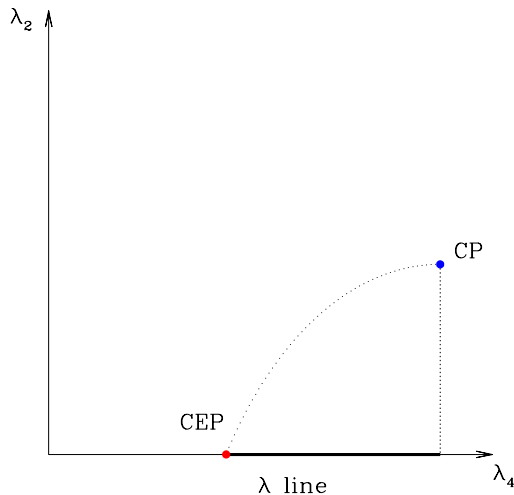
The next example that we have is related to the spontaneous breaking of scale invariance in a three dimensional bosonic theory. Such a theory describes the mixing of  $He_3$  and  $He_4$ , (see [26] and references there in).

The most general Lagrangian describing such a system is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \lambda_2 (\phi)^2 + \frac{\lambda_4}{4N} (\phi)^4 + \frac{\lambda_6}{N^2} (\phi)^6, \quad (73)$$

and it can be treated at  $d = 3 - \varepsilon$ . The system has two order parameters  $\langle (\phi)^2 \rangle$ , and  $\langle \phi \rangle$ .

In a classical analysis performed for  $d = 3 - \varepsilon$ , when the sign of  $\lambda_2$  changes then  $\langle \phi \rangle$  is produced. However,  $\langle (\phi)^2 \rangle \neq 0$  even for  $\lambda_2 > 0$ , which is exemplified by the diagram shown in fig. 16.



**Fig. 16.** The Phase Diagram of a  $d = 3 - \varepsilon$  Conformal Theory, in three dimensions the  $CP$  and  $CEP$  points coincide to produce a flat direction, see [26] for more details

When one goes to three dimensions, the point which is denoted by  $CP$ , which is a critical point, and the point  $CEP$  which is the critical end point, do actually meet together and lead to a very interesting structure. Going directly to  $d = 3$ , one can write down the  $O(N)$  vector model written below

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \lambda_2 (\phi)^2 + \frac{\lambda_4}{4!N} (\phi)^4 + \frac{\lambda_6}{6!N^2} (\phi)^6. \quad (74)$$

It should be emphasized that everything said depends on the very specific manner of taking the limit. One first keeps the cutoff  $\Lambda$  fixed and takes  $N \rightarrow \infty$ , by performing a functional integral or selecting a subset of diagrams, and only then does one remove the cutoff, sending it to infinity, setting the renormalized quadratic and quartic couplings to zero. Such a system turns out to be not only classically conformally invariant, but also quantum mechanically, having a vanishing beta function [27]. We next elaborate on such systems.

Let us now review some more known facts about the three dimensional theory once a classically marginal operator,  $(\phi^2)^3$ , is added [27]. For any finite value of  $N$ , the coupling  $g_6$  of this operator is infrared free quantum mechanically, as the marginal operator gets a positive anomalous dimension already at one loop. This implies that the theory is only well defined for zero value of the coupling of this operator. In the presence of a cutoff interacting particles have mass of the order of the cutoff. At its tri-critical point the  $O(N)$  model in three dimensions is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{6N^2} g_6 (\phi^2)^3, \quad (75)$$

where the fields  $\phi$  are in the vector representation of  $O(N)$ .  
In the limit  $N \rightarrow \infty$  [27]

$$\beta_{g_6} = 0. \quad (76)$$

$1/N$  corrections break conformality. In the large  $N$  limit, then,  $g_6$  is a modulus. It turns out that there is no spontaneous breaking of the  $O(N)$  symmetry and it is instructive to write the effective potential in terms of an  $O(N)$  invariant field,

$$\sigma = \phi^2. \quad (77)$$

The effective potential is [26]:

$$V(\sigma) = f(g_6) |\sigma|^3, \quad (78)$$

where:

$$f(g_6) = g_c - g_6 \quad (79)$$

with

$$g_c = (4\pi)^2. \quad (80)$$

The system has various phases. For values of  $g_6$  smaller than  $g_c$ , i.e. when  $f(g_6)$  is positive, the system consists of  $N$  massless non-interacting  $\phi$  particles. These particles do not interact in the infinite  $N$  limit; thus, correlation functions do not depend on  $g_6$ . For the special value  $g_6 = g_c$ ,  $f(g_6)$  vanishes and a flat direction in  $\sigma$  opens up: the expectation value of  $\sigma$  becomes a modulus. For a zero value of this expectation value, the theory continues to consist of  $N$  massless  $\phi$  fields. For any non-zero value of the expectation value the system has  $N$  massive  $\phi$  particles. All have the same mass due to the unbroken  $O(N)$  symmetry. Scale invariance is broken spontaneously though the vacuum energy still vanishes. The Goldstone boson associated with the spontaneous breaking of scale invariance, the dilaton, is massless and identified as the  $O(N)$  singlet field  $\delta\sigma \equiv \sigma - \langle\sigma\rangle$ . All the particles are non-interacting in the infinite  $N$  limit. This theory is not conformal: in the infrared limit, it flows to another theory containing a single, massless,  $O(N)$ -singlet particle. For larger values of  $g_6$  the exact potential is unbounded from below. The system is unstable (in the supersymmetric case the potential is bounded from below and the larger  $g_6$  structure is similar to the smaller  $g_6$  structure [28]). This case is useful to illustrate the fate of some gravitational instabilities [29]. Actually this instability is an artifact of the dimensional regularization used above, which does not respect the positivity of the renormalized field  $\sigma$ . In any case a more careful analysis [27] shows that the apparent instability reflects the inability to define a renormalizable interacting theory. All the masses are of the order of



**Table 1.** Marginal Perturbations of the  $O(N)$  Model

$f(g_6)$	$ \langle\sigma\rangle $	S.B.	masses	$V$
$f(g_6) > 0$	0	No	0	0
$f(g_6) = 0$	0	No	0	0
$f(g_6) = 0 \neq 0$		Yes	Massless dilaton, $N$ particles of equal mass	0
$f(g_6) < 0$	$\infty$	Yes, but ill defined	Tachyons or masses of order the cutoff	$-\infty$

the cutoff, and there is no mechanism to scale them down to low mass values. In other words, the theory depends strongly on its UV completion.

This is summarized in Table 1.

There, S.B. denotes spontaneous symmetry breaking of scale invariance and  $V$  is the vacuum energy. For  $f(g_6) < 0$  the theory is unstable. Note that the vacuum energy always vanishes whenever the theory is well-defined.

When  $\langle\sigma\rangle \neq 0$  and the scale invariance is spontaneously broken, one can write down the effective theory for energy scales below  $\langle\sigma\rangle$ , and integrate out the degrees of freedom above that scale. The vacuum energy remains zero however, and it is not proportional to  $\langle\sigma\rangle^3$ , [26], [31], [32], [32], [39], [30], as might be expected naively.

For completeness we note that by adding more vector fields one has also phases in which the internal global  $O(N)$  symmetry is spontaneously broken.

An example is the  $O(N) \times O(N)$  model [32] with two fields in the vector representation of  $O(N)$ , with Lagrangian:

$$\mathcal{L} = \partial_\mu \phi_1 \cdot \partial^\mu \phi_1 + \partial_\mu \phi_2 \cdot \partial^\mu \phi_2 + \lambda_{6,0}(\phi_1^2)^3 + \lambda_{4,2}(\phi_1^2)^2(\phi_2^2) + \lambda_{2,4}(\phi_1^2)(\phi_2^2)^2 + \lambda_{0,6}(\phi_2^2)^3. \quad (81)$$

Again, the  $\beta$  functions vanish in the strict  $N \rightarrow \infty$  limit. There are now two possible scales, one associated with the breakdown of a global symmetry and another with the breakdown of scale invariance. The possibilities are summarized by the table below:

$O(N)$	$O(N)$	scale	massless	massive	$V$
+	+	+	all	none	0
-	+	-	$(N-1)\pi's, D$	$N, \sigma$	0
+	-	-	$(N-1)\pi's, D$	$N, \sigma$	0
-	-	-	$2(N-1)\pi's, D$	$\sigma$	0

Again in all cases the vacuum energy vanishes. Assume a hierarchy of scales where the scale invariance is broken at a scale much above the scale at which the  $O(N)$  symmetries are broken. One would have argued that one

would have had a low energy effective Lagrangian for the massless pions and dilaton with a vacuum energy given by the scale at which the global symmetry is broken. This is not true, the vacuum energy remains zero. This system has a critical surface, on one patch the deep infrared theory contains only one massless particle: an  $O(N) \times O(N)$  singlet. For the other patches the deep infrared theory is described by  $O(N)$  massless particles, most of which are not  $O(N)$  singlets. The dimension of the surface for which spontaneous symmetry breakings occur is smaller than that of the full space of parameters. I will not consider spontaneous symmetry breaking as fine tuning.

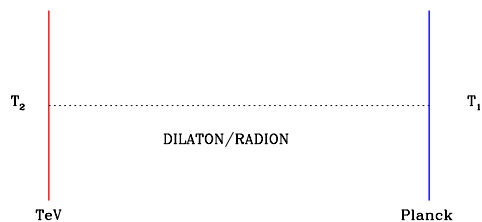
In general, effective field theories should have all possible symmetries of the underlying theory, whether they are realized linearly or non-linearly. In finite scale invariant theories the vacuum energy  $E_{vac}$  should be determined by all scales and symmetries involved. It should have the same value, (zero in this case), in all phases of the system whether or expectation values are formed. This punches a hole in Zaldowitch like arguments [33], and offers a different view on the gravity of the Cosmological Constant problem [34]. If the theory has a global scale invariance, which is spontaneously broken, it will produce a dilaton. The question is: Where is the dilaton? The dilaton should be a massless field. Several authors, [35],[36] tried to check the possibilities that the dilatons might exist, noting that the dilaton must be a massless Goldstone boson. Under certain assumptions, one finds out that actually in certain models having a massless dilaton would not violate experimental data. Perhaps it even predicts deviations of the equivalence principle from Galileo famous experiment just below the present experimental sensitivity  $\delta a/a \sim 10^{-12}$ .

This is done under the assumption that the dilaton couples in the following universal fashion

$$\mathcal{L} = F(\Phi) \left( R - F^2 + 2[\nabla^2 \Phi - (\nabla \Phi)^2] \right) . \quad (83)$$

It could also happen that the dilaton gets swallowed in some Higgs like mechanism. One should also mention that if kinematically a finite scale-invariant is forced by some super-selection rule (such as having a non trivial monopole number [40]) into a certain solitonic sector, then the rest energy of the system should be accounted for and the vacuum energy will be slightly lifted from zero. Let us finish this section by noticing an amusing thing, there are various solutions that go under the name of Randall and Sundrum [37]. One of the constructions contains two types of branes, near the boundary of the space there is a Planck brane with tension  $T_1$ , which is fine tuned so to have zero Cosmological Constant. Then at a certain distance, very deep inside the bulk theory, one places the TeV brane, it has negative tension and the tension is again fine tuned, so that the Cosmological Constant vanishes also on that brane.

The two branes are separated by some distance which in [38] is associated to massless particle, which is the dilaton or the radion, (see fig. 17).



**Fig. 17.** Planck brane and Tev brane

In principle, there are circumstances where this distance is not fixed, and there are several possible situations whose outcome is very similar to that one discussed in the  $d = 3$  conformal theory. If the sum of the tensions  $T_1 + T_2$  is arranged to vanish, then the system behaves as a spontaneously broken system, the magnitude of the vev of the field is the distance between the two branes.

If  $T_1 + T_2 > 0$  the two branes actually are attracted to sit one on top of the other, and when  $T_1 + T_2 < 0$  the branes repel, the system is unstable and as a result one of the branes is exiled to infinity.

These three examples are in full correspondence with the conditions on the coefficients of the  $(\phi)^6$  theory that we discussed above. The difference between the two theories, and an important difference is that in case of the  $(\phi)^6$  theory we are certain that in the large  $N$  limit, the theory is indeed finite quantum mechanically.

For the case of  $(\phi)^4$  we don't have such an assurance, and it would be nice to find a system for which we are guaranteed to be finite also quantum mechanically, which exhibits the same type of behavior.

## 5.1 Conclusions

- Spontaneous breaking of translational and rotational symmetry are possible. It fits data for many phases of matter, and it may have a manifestation in the dynamics of compactification.
- Conformal/Scale invariant theories which are stable but have no ground states indicate a new mechanism of breaking time translational invariance as well as supersymmetry.
- A finite scale invariant theory has the same (vanishing) vacuum energy in all its phases.
- It is a great privilege to recognize Gabriele's outstanding contributions.

## Acknowledgements

The author thanks Matteo Cardella for various discussions on this manuscript. The author wishes to thank his various collaborators on these subjects, especially W. Bardeen, S. Elitzur, M. Einhorn, A. Forge, A. Giveon, M. Porrati, A. Schwimmer, AND G. Veneziano.

## References

1. S. Fubini and E. Rabinovici, Nucl. Phys. B **245** (1984) 17

2. J. Hughes and J. Polchinski, Nucl. Phys. B **278** 147 (1986).
3. L.D.Landau, Phys. Z. Soviet II **26**, 545 (1937)
4. S. Elitzur, A. Forge and E. Rabinovici, "Some global aspects of string compactifications," Nucl. Phys. B **359** (1991) 581.
5. C. Baym, H.A. Bethe, and C.J. Pethick, Nucl Phys A **175**, 25 (1971)
6. S. Alexander: Symmetries and Broken Symmetries in Condensed Matter Physics, ed. N. Boccara (IDSET Paris,1981) p.141, and references therein.
7. A.B. Zamolodchikov, Sov. Phys. JETP Lett. **43**, 730 (1986)
8. A.W.W. Ludwig and J.L. Cardy, Nucl. Phys. B**285** [FS19], 687 (1987)
9. S. Elitzur, E. Rabinovici and G. Sarkissian, Nucl. Phys. B **541**, 246 (1999)
10. I. Affleck and A. W. W. Ludwig, Phys. Rev. Lett. **67**, 161 (1991)
11. S. L. Shatashvili, Alg. Anal. **6**, 215 (1994)
12. D. Friedan and A. Konechny, Phys. Rev. Lett. **93**, 030402 (2004) [arXiv:hep-th/0312197].
13. E. Witten, Phys. Rev. D **47**, 3405 (1993)
14. J. A. Harvey, D. Kutasov and E. J. Martinec, "On the relevance of tachyons," arXiv:hep-th/0003101.
15. V. de Alfaro, S. Fubini and G. Furlan, Nuovo Cim. A **34** (1976) 569
16. V. P. Akulov and A. I. Pashnev, Theor. Math. Phys. **56** (1983) 862 [Teor. Mat. Fiz. **56** (1983) 344]
17. P. Claus, M. Derix, R. Kallosh, J. Kumar, P. K. Townsend and A. Van Proeyen, Phys. Rev. Lett. **81** (1998) 4553
18. R. Kallosh, "Black holes and quantum mechanics," arXiv:hep-th/9902007.
19. E. D'Hoker and R. Jackiw, Phys. Rev. Lett. **50**, 1719 (1983)
20. C. W. Bernard, B. Lautrup and E. Rabinovici, Phys. Lett. B **134**, 335 (1984)
21. T. R. Taylor, G. Veneziano and S. Yankielowicz, Nucl. Phys. B **218**, 493 (1983)
22. I. Affleck, M. Dine and N. Seiberg, Nucl. Phys. B **241**, 493 (1984).
23. Sergio Fubini, "A New Approach To Conformal Invariant Field Theories," CERN-TH-2129, Feb 1976. 46pp. Published in Nuovo Cim. **A34**: 521,1976.
24. J. Polchinski, Nucl. Phys. **303**, 226 (1988)
25. S. Elitzur, A. Giveon, E. Rabinovici, A. Schwimmer and G. Veneziano, Nucl. Phys. B **435**, 147 (1995)
26. D. J. Amit and E. Rabinovici, Nucl. Phys. B **257**, 371 (1985)
27. W. A. Bardeen, M. Moshe and M. Bander, Phys. Rev. Lett. **52**, 1188 (1984).
28. W. A. Bardeen, K. Higashijima and M. Moshe, Nucl. Phys. B **250**, 437 (1985)
29. S. Elitzur, A. Giveon, M. Porrati and E. Rabinovici, JHEP **0602**, 006 (2006)
30. D. S. Berman and E. Rabinovici, "Supersymmetric gauge theories," arXiv:hep-th/0210044.
31. M. B. Einhorn, G. Goldberg and E. Rabinovici, Nucl. Phys. B **256**, 499 (1985).
32. E. Rabinovici, B. Saering and W. A. Bardeen, Phys. Rev. D **36**, 562 (1987).
33. Ya. B. Zeldovich, Sov. Phys. Uspekhi **11** (1968) 381.
34. S. Weinberg, Rev. Mod. Phys. **61** (1989)
35. T. Damour and A. M. Polyakov, Nucl. Phys. B **423**, 532 (1994)
36. T. Damour, F. Piazza and G. Veneziano, Phys. Rev. D **66**, 046007 (2002)
37. L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999)
38. R. Rattazzi and A. Zaffaroni, JHEP **0104**, 021 (2001)
39. E. Rabinovici in W. M. Alberico and S. Sciuto: Symmetry And Simplicity In Physics. In: *Proceedings, Symposium On The Occasion Of Sergio Fubini's 65th Birthday*, Turin, Italy, February 24-26, 1994, Singapore (World Scientific 1994) p 220.
40. E. Rabinovici, unpublished.